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Numerical Calculation of Economic Uncertainty by Intervals and Fuzzy Numbers

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Abstract

This paper emphasizes that numerically correct calculation of economic uncertainty with intervals and fuzzy numbers requires implementation of global optimization techniques in contrast to straightforward application of interval arithmetic. This is demonstrated by both a simple case from managerial economics as well as a real life railway reconstruction project. Based on identical uncertain input data the difference between the probabilistic and the possibilistic approach is highlighted, the latter producing a substantial larger degree of numerical uncertainty due to its non-statistical nature.

Keywords: economic uncertainty, intervals, fuzzy numbers, probability, possibility

1 Introduction

The notion of risk and uncertainty being relevant for economic analysis was suggested by Knight [13] and the concepts were incorporated into economic theory by von Neumann and Morgenstern [17] who developed a rational foundation and rules for decision making according to expected utility, see also [9]. Thus, the traditional approach to representation of uncertainty in economic theory is that of probabilities. An uncertain variable may be represented by a probability distribution reflecting either the objective nature of the variable or the subjective belief of the agent. The most common objectivist position argues that the probability of a particular event in a particular trial is the relative frequency of occurrence of that event in an infinite sequence of similar trials. Obviously, the idea of infinite repetition is referring to an idealized laboratory experimental situation like rolling an ideal dice an infinite number of times. How then is one to comprehend the probability of one-of-a-kind-events, such as the probability of a quote leading to an order or the cost incurred to build a new opera house?

Consequently, there have been many objections to this view of probability arguing that randomness is not an objectively measurable phenomenon but rather a knowledge phenomenon. Thus, probabilities are rather an epistemological and not an ontological issue. This epistemic or knowledge view of probability can be traced back to [1, 14]. More recently, Ramsey [19] asserted that probability is related to the knowledge possessed by a particular individual and thus probability represents personal belief rather than objective knowledge. Probability theory and statistics today represent a well-established mathematical theory with clear axioms and has reached an advanced stage of development. However, criticism has been raised towards probability theory as being a too normative framework to take all the aspects of uncertain judgment into account, see e.g. [7]. In this paper, we will focus on alternative methods of modelling economic uncertainty like the interval representation and the fuzzy number representation.

The interval representation [15, 16] is particularly well suited to a situation where the knowledge of an uncertain parameter is limited to knowing its minimum and maximum value whereas nothing else is known. Based upon a mathematical theory of interval analysis this approach has shown to be useful in keeping track of worst and best cases in economic analyses and thus contribute to improved decision processes, see e.g. [20, 21]. Zadeh [25] introduced the concept of fuzzy set for the purpose of modelling the imprecision and ambiguity of ordinary language. It is based on the concept of possibility rather than probability and translates natural language expressions into the mathematical formalism of possibility measures. It is generally recognized that possibility is distinct from probability. As mentioned earlier, probabilities can be interpreted as relative frequencies or, more generally, uncertain knowledge or

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belief of a statistical nature. In contrast, possibility relates to the degree of feasibility and ease of attainment or imprecise knowledge.

Particularly, this paper will deal with some computational aspects of uncertainty representation by intervals and fuzzy numbers. It is the author's experience that these aspects are widely neglected, probably due to lack of communication and interaction between professional communities each one dealing with separate issues. In Section 2 a simple case from managerial economics is presented in order to serve as a reference case for the subsequent numerical calculations. In Section 3, the basics of interval analysis are briefly stated and subsequently applied to the simple case in order to demonstrate the difference between straightforward interval arithmetic and global optimization considering both independent and interdependent uncertain input variables as well as uncertain model characteristics. Section 4 focuses on fuzzy numbers, particularly the procedure of applying global optimization on α-cuts in order to get correct results. In Section 5, a real life example of estimating the cost of a railway reconstruction project is treated by representing uncertainty by triangular and trapezoidal fuzzy numbers. Finally, in Section 6, comparisons are made with Monte Carlo simulations in order to demonstrate the quite substantial differences generated by the possibility and probability approaches, respectively. Section 7 contains a conclusion.

2 A Simple Case from Managerial Economics

In order to demonstrate the principles of numerical modelling of uncertainty by intervals and fuzzy numbers presented in this paper we introduce a simple yet instructive case from managerial economics. Consider a company selling one product at price p and quantity q into a market. For the sake of simplicity, we may assume fixed cost to be zero. The turnover TR is

$$TR = TR(p,q) = p \cdot q, \tag{1}$$

and the variable cost VC is given by

$$VC = VC(q) = 20 \cdot q. \tag{2}$$

Then the profit π is

$$\pi = \pi(p,q) = TR(p,q) - VC(q) = p \cdot q - 20 \cdot q. \tag{3}$$

Clearly, we have an ordinary non-fuzzy model for the profit as a function of price and quantity. Not knowing better, it is assumed that quantity and price are independent variables. The profit is mapped in Table 1.

We want to find the price p* and quantity q* that gives the maximum profit π*. It is seen from (3) that the profit is a monotone function of q and p, which means that no maximum of π exists unless the variables are constrained. If for example $p \le 65$ and $q \le 450$, then p* = 65, q* = 450, and π* = 20.250 (for comparison see Table 1).

Table 1: Mapping of profit function $\pi = \pi(p,q)$, independent price and quantity

p,q independent $\pi(p,q) = p \cdot q - 20 \cdot q$		Quantity q						
		250	300	350	400	450	500	550
Price p	75	13.750	16.500	19.250	22.000	24.750	27.500	30.250
	70	12.500	15.000	17.500	20.000	22.500	25.000	27.500
	65	11.250	13.500	15.750	18.000	20.250	22.500	24.750
	60	10.000	12.000	14.000	16.000	18.000	20.000	22.000
	55	8.750	10.500	12.250	14.000	15.750	17.500	19.250
	50	7.500	9.000	10.500	12.000	13.500	15.000	16.500
	45	6.250	7.500	8.750	10.000	11.250	12.500	13.750

Next, we recognize that price and quantity are interdependent variables due to market conditions. Assuming that they are connected by the demand function

$$p = p(q) = 100 - 0{,}1 \cdot q \;\text{ or }\; q = q(p) = 1.000 - 10 \cdot p, \tag{4}$$

we get for the turnover

$$TR = TR(p) = -10 \cdot p^2 + 1.000 \cdot p, \tag{5}$$

the variable cost

$$VC = VC(p) = -200 \cdot p + 20.000, \tag{6}$$

and the profit margin

$$\pi = \pi(p) = TR(p) - VC(p) = -10 \cdot p^2 + 1.200 \cdot p - 20.000. \tag{7}$$

Mapping of the profit function is redone under the new conditions and the results are shown in Table 2.

We determine the maximum profit of (7) by the optimality criterion that marginal cost MC be equal to marginal turnover MR which gives the results (compare with Table 2)

$$p^* = 60,\ q^* = 400, \text{ and } \pi^* = 16.000. \tag{8}$$

Table 2: Mapping of profit function $\pi = \pi(p,q)$, interdependent price and quality

$q = 1.000 - 10 \cdot p$ $\pi(p,q) = p \cdot q - 20 \cdot q$		Quantity q						
		250	300	350	400	450	500	550
Price p	75	13.750	-	-	-	-	-	-
	70	-	15.000	-	-	-	-	-
	65	-	-	15.750	-	-	-	-
	60	-	-	-	16.000	-	-	-
	55	-	-	-	-	15.750	-	-
	50	-	-	-	-	-	15.000	-
	45	-	-	-	-	-	-	13.750

In the following, we shall refer to the above examples for explanations of computational details and special features.

3 Uncertainty Modelling Using Intervals

3.1 Basics of Interval Analysis

Following Moore [16] and Caprani, Madsen, and Nielsen [2], we define a real interval number as an ordered pair [a; b] of real numbers with $a \le b$. It may also be defined as an ordinary set of real numbers x such that $a \le x \le b$, or

$$[a; b] = \{x \mid a \le x \le b\}. \tag{9}$$

If the basic arithmetic operations addition, subtraction, multiplication, and division are denoted by the symbol # we can define operations on two intervals $\mathbf{I}_1 = [a_1; b_1]$ and $\mathbf{I}_2 = [a_2; b_2]$ based on the set-theoretic formulation:

$$\mathbf{I}_1 \# \mathbf{I}_2 = \{x \# y \mid a_1 \le x \le b_1,\ a_2 \le y \le b_2\}. \tag{10}$$

For basic operations on the intervals $\mathbf{I}_1$ and $\mathbf{I}_2$ we get the resulting interval $\mathbf{I} = [a; b]$ by the formulas

$$\begin{aligned}
\mathbf{I} &= \mathbf{I}_1 + \mathbf{I}_2 = [a_1+a_2;\ b_1+b_2],\\
\mathbf{I} &= \mathbf{I}_1 - \mathbf{I}_2 = [a_1-b_2;\ b_1-a_2],\\
\mathbf{I} &= \mathbf{I}_1 \cdot \mathbf{I}_2 = [\min(a_1 \cdot a_2, a_1 \cdot b_2, b_1 \cdot a_2, b_1 \cdot b_2);\ \max(a_1 \cdot a_2, a_1 \cdot b_2, b_1 \cdot a_2, b_1 \cdot b_2)],\\
\mathbf{I} &= \mathbf{I}_1 / \mathbf{I}_2 = [\min(a_1/a_2, a_1/b_2, b_1/a_2, b_1/b_2);\ \max(a_1/a_2, a_1/b_2, b_1/a_2, b_1/b_2)],\ 0 \notin [a_2; b_2].
\end{aligned} \tag{11}$$

It can be shown that the four basic interval operations are inclusion monotonic, and that addition and multiplication are commutative and associative, [2]. However, the distributive rule is not valid in general. Instead, the so-called sub-distributivity holds

$$\mathbf{I}_1 \cdot (\mathbf{I}_2 + \mathbf{I}_3) \subseteq \mathbf{I}_1 \cdot \mathbf{I}_2 + \mathbf{I}_1 \cdot \mathbf{I}_3. \tag{12}$$

From a rational real valued function F of n real valued variables

$$F = F(x_1, x_2, \ldots, x_n), \tag{13}$$

we can create the interval extension function as an interval function $\mathbf{F}^I$ of n intervals

$$\mathbf{F}^I = \mathbf{F}^I(\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n), \tag{14}$$

simply by replacing the real operators by interval operators and the real variables by intervals.

A rational function can be formulated in many ways whereas the same reformulations cannot be done for interval expressions due to the invalidity of the distributive rule. This implies that different formulations of a rational function will lead to different interval extension functions and thus to different interval results. In the case of F being a monotonic function within the entire range of the input variables the minimum and maximum of $\mathbf{F}^I$ as an interval can simply be found among the function values F at the extreme points of the variables. In the general case of F being non-monotonic or variables appearing more then once, the calculation of $\mathbf{F}^I$ as an interval is non-trivial, which is demonstrated by the following example.

Example 1: Based on the function $F = x \cdot (1 - x)$ the interval function $\mathbf{F}^I = \mathbf{I} \cdot (1 - \mathbf{I})$, $\mathbf{I} = [0; 1]$ is calculated. Straightforward application of formulas from (11) gives the result $\mathbf{F}^I = [0; 1]$ whereas the correct result obtained by global optimization is [0; 0,25].

In this paper, the term "correct" is used to indicate the narrowest possible interval that can be calculated for an uncertain variable. Generally, to obtain this, iterative global optimization methods have to be used, see e.g. [8, 12]. In order to obtain correct results (as in the above example) to an accuracy specified by the user, interval calculations in this paper are carried out using the Interval Solver 2000 program [10, 11], as an add-in module to MS-Excel 2000. An overall absolute and relative precision of 10^{-6} has been applied.

Correct calculation of interval functions allows for strong statements about the uncertainties involved. *Firstly*, you can say that provided all uncertain input variables stay within their minimum and maximum values, the uncertain output function will stay within its minimum and maximum values. *Secondly*, the uncertain output function will not attain any value that is not a function value of some combination of the uncertain input values (within their minimum and maximum values).

3.2 Uncertainty Analysis with Independent and Dependent Variables

3.2.1 Independent Input Variables

Using interval arithmetic

To calculate the uncertain profit we first use the formulas of interval arithmetic (11) by three different ways of calculation. As an example look at the independent and uncertain quantity and price

$$\mathbf{p} = [55; 65],\ \mathbf{q} = [350; 450]. \tag{15}$$

Firstly, we use the turnover and variable cost as intermediate variables. By (1) we get for the uncertain turnover **TR** and by (2) for the uncertain variable cost **VC**

$$\mathbf{TR} = [55; 65]\cdot[350; 450] = [19.250; 29.250],\ \ \mathbf{VC} = 20\cdot[350; 450] = [7.000; 9.000]. \tag{16}$$

Then by (3) we get for the uncertain profit

$$\boldsymbol{\pi} = \mathbf{TR} - \mathbf{VC} = [10.250; 22.250]. \tag{17}$$

However, the above calculation produces a too wide interval for the profit. The reason for this is, that in the expression (3) the variable q appears twice thus allowing the quantity q used to calculate **TR** to be different from the quantity used to calculate **VC**.

Secondly, the rightmost form of (3) is used, giving

$$\boldsymbol{\pi} = [350; 450]\cdot[55; 65] - 20\cdot[350; 450] = [10.250; 22.250], \tag{18}$$

which is identical to (17) because the arithmetic operations are identical.

Thirdly, (3) is rearranged before the interval calculations are carried out:

$$\pi = \pi(p,q) = q\cdot(p - 20). \tag{19}$$

We then get for the uncertain profit

$$\boldsymbol{\pi} = [350; 450]\cdot([55; 65] - 20) = [350; 450]\cdot[35; 45] = [12.250; 20.250], \tag{20}$$

which is a somewhat narrower interval than (17) and (18) because each variable is appearing only once in (19). Actually, (20) is the correct result, compare with Table 1.

Using global optimization

Next, we calculate the uncertain profit by global optimization using Interval Solver 2000. With the same input variables (15) and intermediate variables **TR** and **VC** we get from (3)

$$\mathbf{TR} = [19.250; 29.250],\ \mathbf{VC} = [7.000; 9.000],\ \boldsymbol{\pi} = \mathbf{TR} - \mathbf{VC} = [10.250; 22.250], \tag{21}$$

which is identical to (17) and (18) for the same reason as mentioned above. With the same input variables and by way of formulas (18) and (19) we get the result

$$\mathbf{p} = [55; 65],\ \mathbf{q} = [350; 450],\ \boldsymbol{\pi} = [12.250; 20.250], \tag{22}$$

which is seen to be the correct result identical to (20). The results are summarized in Table 3.

Table 3: Resume of uncertainty analyses, independent price **p** = [55; 65] and quantity **q** = [350; 450]

Calculation Method	$\boldsymbol{\pi} = \mathbf{TR} - \mathbf{VC}$	$\boldsymbol{\pi} = \mathbf{p}\cdot\mathbf{q} - 20\cdot\mathbf{q}$	$\boldsymbol{\pi} = \mathbf{q}\cdot(\mathbf{p} - 20)$
Interval Arithmetic	[10.250; 22.250]	[10.250; 22.250]	[12.250; 20.250]
Global Optimization	[10.250; 22.250]	[12.250; 20.250]	[12.250; 20.250]

3.2.2 Interdependent Input Variables

Using interval arithmetic

To perform an uncertainty analysis around a given price p = 60 we set

$$\mathbf{p} = [55; 65]. \tag{23}$$

The profit is calculated in two different ways using interval arithmetic (11). *Firstly*, the profit is calculated by intermediate variables **TR** and **VC** according to (5) and (6), respectively

$$\mathbf{TR} = -\,10\cdot[55; 65]^2 + 1.000\cdot[55; 65] = [12.750; 34.750], \tag{24}$$

and

$$\mathbf{VC} = -\,200\cdot[55; 65] + 20.000 = [7.000; 9.000], \tag{25}$$

which gives

$$\boldsymbol{\pi} = \mathbf{TR} - \mathbf{VC} = [3.750; 27.750]. \tag{26}$$

Secondly, the profit is calculated by the formula

$$\boldsymbol{\pi} = -\ 10\cdot\mathbf{p}^2 + 1.200\cdot\mathbf{p} - 20.000, \tag{27}$$

yielding

$$\boldsymbol{\pi} = -\ 10\cdot[55; 65]^2 + 1.200\cdot[55; 65] - 20.000 = [3.750; 27.750]. \tag{28}$$

The results found above are identical and way out of order, since the correct result of the uncertain profit is [15.750; 16.000], compare with (8) and Table 2.

Using global optimization

Global optimization is used to calculate the uncertain profit by means of the formulas (23)-(26) and (27) and the results are shown in Table 4. It may be observed that correct results are produced, except in the case where the usage of intermediate variables **TR** and **VC** allows for a too wide interval.

Table 4: Resume of uncertainty analyses, interdependent price **p** = [55; 65] and quantity **q** = 1.000 – 10**·p**

Calculation Method	$\boldsymbol{\pi} = \mathbf{TR} - \mathbf{VC}$	$\boldsymbol{\pi} = -\ 10\cdot\mathbf{p}^2 + 1.200\cdot\mathbf{p} - 20.000$
Interval Arithmetic	[3.750; 27.750]	[3.750; 27.750]
Global Optimization	[13.750; 17.750]	[15.750; 16.000]

The difference between the correct interval and the more wide intervals in Table 4 is considerable. It emphasizes the fact that global optimization is indispensable when calculating with intervals. Looking at the correct results in Table 4 (lower right cell), the decision maker knows that if he decides to set the price between 55 and 65 his profit will come out between 15.750 and 16.000. However, he does not know what specific profit a particular price will produce. Left with this uncertainty he might as well choose the price 55 instead of 65. Without prior knowledge of the information given in Table 2, he will never get to know that the price 60 will result in the maximum profit according to the non-fuzzy model.

3.3 Analysis with Uncertain Model

The equations (1)-(3) constitute the non-fuzzy model of Section 2 in the case of independent input variables p and q and output variable π. When p and q are interdependent by the demand function the model is extended with equation (4). When the input parameters are uncertain variables **p** and **q**, the profit output variable $\boldsymbol{\pi}$ also becomes uncertain.

The model of Section 2 is easily extended into an uncertain model simply by substituting the constants in (2) and (4) with intervals representing the uncertainty. For the uncertain variable cost (2) then becomes

$$\mathbf{VC} = \mathbf{VC}(\mathbf{q}) = \mathbf{a}\cdot\mathbf{q}, \tag{29}$$

and the uncertain demand function (4) becomes

$$\mathbf{q} = \mathbf{q}(\mathrm{p}) = (\mathrm{p} - \mathbf{p}_0)/\mathbf{p}_1. \tag{30}$$

Table 5: Mapping of demand function $q = q(p) = (p - p_0)/p_1$

Quantity q $q = (p - p_0)/p_1$, p = 60		p_1						
		–0,115	–0,110	–0,105	–0,100	–0,095	–0,090	–0,085
p_0	115	478,3			550,0			647,1
	110		454,5		500,0		555,6	
	105			428,6	450,0	473,7		
	100	347,8	363,6	381,0	400,0	421,1	444,4	470,6
	95			333,3	350,0	368,4		
	90		272,7		300,0		333,3	
	85	217,4			250,0			294,1

Table 5 contains a mapping of the demand function showing a quite substantial sensitivity of the quantity q as a function of the constants p_0 and p_1. A deviation from the nominal value p_0 = 100 of ± 10% gives a deviation of q of ± 25%. Likewise, a deviation of ± 10% from the nominal value of p_1 = –0,1 gives a deviation of q of +11% and –9%.

An expression of the uncertain profit $\boldsymbol{\pi}$ calculated at the certain price p is easily derived from (29) and (30)

$$\boldsymbol{\pi} = \mathrm{p}^2/\mathbf{p}_1 - \mathrm{p}\cdot(\mathbf{p}_0/\mathbf{p}_1 + \mathbf{a}/\mathbf{p}_1) + \mathbf{a}\cdot\mathbf{p}_0/\mathbf{p}_1. \tag{31}$$

Calculating (31) with the price p = 60 and uncertain constants $\mathbf{p}_0$ = [90; 110], $\mathbf{p}_1$ = [–0,11; –0,09], and **a** = [18; 22], by using global optimization, we get the correct result $\boldsymbol{\pi}$ = [10.364; 23.333].

In conclusion, introducing an uncertainty of ± 10% in the constants defining the model produces an uncertainty of +46% and –35% in the resulting profit. This means, that although the decision maker sets the price to 60 in order to maximize profit in accordance with the theoretical model without uncertainty he might experience a dramatically higher as well as lower profit. This is mainly due to the uncertain demand function that may result in both higher and lower quantities of sale.

What we have seen in Section 3.3 is an example of a model with uncertainty and an input parameter without uncertainty producing an uncertain output. In Sections 3.1 and 3.2 a model without uncertainty with an uncertain input also producing an uncertain output. Of course, a combination of uncertain model and uncertain input also will also produce an uncertain output. A detailed discussion of some of these aspects can be found in [3].

4 Uncertainty Modelling using Fuzzy Numbers

4.1 Fuzzy Numbers and Intervals

Following the seminal paper by Zadeh [25] a fuzzy set A in X where X is a space of points (objects) with a generic element of X denoted by x, i.e. X = {x}, is characterized by a membership function $f_A(x)$ which associates with each point in X a real number in the interval [0; 1]. The value of the membership function $f_A(x)$ at x represents the "grade of membership" of x in A. Thus the closer the value of $f_A(x)$ to unity, the higher the grade of membership of x in A. Note that when A is an ordinary set, i.e. non-fuzzy, the membership function can take only two values 0 and 1.

$$A = \{(x, f_A(x)) \mid x \in X\}. \tag{32}$$

It is also useful to define the ordinary (non-fuzzy) set A_α as the α-cut of A:

$$A_\alpha = \{x \in X \mid f_A(x) \geq \alpha, 0 \leq \alpha \leq 1\}. \tag{33}$$

In this paper, we are mainly interested in the concept of fuzzy numbers as a means of representing uncertain or fuzzy information [5, 6]. In addition to the simplest fuzzy number, namely the interval, we also make use of the triangular fuzzy number [4] [a; c; b], where $a \leq c \leq b$, that can be defined by its membership function:

$$f(x)=\begin{cases}(x-a)/(c-a), & a \leq x \leq c\\ (b-x)/(b-c), & c<x \leq b\\ 0, & \text{otherwise.}\end{cases} \tag{34}$$

We also use the trapezoidal fuzzy number [24] [a; c; d; b], where $a \leq c \leq d \leq b$, defined by the following definition

$$f(x)=\begin{cases}(x-a)/(c-a), & a \leq x \leq c\\ 1, & c<x \leq d\\ (b-x)/(b-d), & d<x \leq b\\ 0, & \text{otherwise.}\end{cases} \tag{35}$$

Mathematical operations on triangular fuzzy numbers can be facilitated by introducing the left $L(\alpha)$ and right $R(\alpha)$ representation of a fuzzy triangular number FT, refer to the α-cut (33):

$$\mathbf{F}^T = [L(\alpha); R(\alpha)], \text{ where } L(\alpha) = a + (c-a)\cdot\alpha \text{ and } R(\alpha) = b + (c-b)\cdot\alpha,\ \alpha \in [0, 1]. \tag{36}$$

For a trapezoidal fuzzy number we have correspondingly:

$$\mathbf{F}^{TR} = [L(\alpha); R(\alpha)], \text{ where } L(\alpha) = a + (c-a)\cdot\alpha \text{ and } R(\alpha) = b + (d-b)\cdot\alpha,\ \alpha \in [0, 1]. \tag{37}$$

Observe that in this notation a fuzzy number is written as an interval with upper and lower bounds depending on α. This means that addition, subtraction, multiplication, and division can be carried out by using interval methods for all values of α. Likewise, for any triangular and trapezoidal function, the resulting triangular and trapezoidal functional values can be calculated and represented by L and R functions using interval methods for all values of α.

Example 2: Based on the real valued function $F = x\cdot(1 - x)$ calculate the corresponding fuzzy function with triangular argument [0; 0,5; 1] and trapezoidal argument [0; 0,3; 0,7; 1]. The correct results have been calculated with global optimization and are shown in Tables 6 and 7.

From the results in Table 6 it can be seen that the function F has a maximum of 0,250 at x = 0,5. Likewise, from Table 7 it can be seen that F has a maximum of 0,250 somewhere between x = 0,3 and 0,7, but not where exactly.

To obtain simpler representations and reduce the number of calculations, triple and quadruple representations of fuzzy variables corresponding to α-cuts 0 and 1 in (33) may be used. In the triangular case the result then is [0; 0,25; 0,25], where the extreme function values is obtained by global optimization on the interval [0; 1] and the interior point is obtained by conventional calculation at x = 0,5. In the trapezoidal case the result is [0; 0,21; 0,25; 0,25], where the "outer interval" [0; 0,25] is obtained by global optimization on the interval [0; 1] and the "inner interval" [0,21; 0,25] by global optimization on the interval [0,3; 0,7].

Table 6: Fuzzy extension of x·(1 – x) calculated with triangular argument [0; 0,5; 1]

α	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
L(α)	0,000	0,047	0,090	0,127	0,160	0,188	0,210	0,227	0,240	0,248	0,250
R(α)	0,250	0,250	0,250	0,250	0,250	0,250	0,250	0,250	0,250	0,250	0,250

Table 7: Fuzzy extension of x·(1 – x) calculated with trapezoidal argument [0; 0,3; 0,7; 1]

α	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
L(α)	0,000	0,029	0,056	0,082	0,106	0,128	0,148	0,166	0,182	0,197	0,210
R(α)	0,250	0,250	0,250	0,250	0,250	0,250	0,250	0,250	0,250	0,250	0,250

4.2 Simple Case with Triangular Fuzzy Numbers

Next we calculate the simple case from Section 2 using profit function (3) with independent triangular fuzzy input parameters **p** = [55; 60; 65], **q** = [350; 400; 450] defined by (34). The resulting membership function is shown in Table 8 for different values of α. It is easily seen that the value α = 0 corresponds to the correct profit interval found in Section 3.2.1 and α = 1 corresponds to the single point calculation of the profit, compare with Table 1. This is one of the important features of the triangular fuzzy number representation of uncertainty: It is easily communicated and understood that the ordinary single point calculation is extended to an interval around the single point representing the uncertain value of the input variable. The resulting triangular fuzzy profit is interpreted as follows: With the given uncertain input variables, the most possible value of the profit is 16.000 and profits outside the interval [12.250; 20.250] are impossible.

Table 8: Uncertain profit $\boldsymbol{\pi}$ (3) by global optimization, triangular fuzzy input **p** = [55; 60; 65], **q** = [350; 400; 450]

α	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
L(α)	12.250	12.603	12.960	13.323	13.690	14.063	14.440	14.823	15.210	15.603	16.000
R(α)	20.250	19.803	19.360	18.923	18.490	18.063	17.640	17.223	16.810	16.403	16.000

Calculations have also been carried out with interdependent triangular fuzzy input parameters corresponding to **p** = [55; 60; 65] and profit function (7), the results are shown in Table 9. For all values of α the correct maximum profit of 16.000 has been found and no profit value outside the interval [15.750; 16.000] is possible, compare with Table 2.

Table 9: Uncertain profit $\boldsymbol{\pi}$ (7) by global optimization, triangular fuzzy input **p** = [55; 60; 65]

α	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
L(α)	15.750	15.797	15.840	15.877	15.910	15.937	15.960	15.977	15.990	15.977	16.000
R(α)	16.000	16.000	16.000	16.000	16.000	16.000	16.000	16.000	16.000	16.000	16.000

Table 10: Uncertain profit $\boldsymbol{\pi}$ (3) by global optimization, trapezoidal independent fuzzy input **p** = [55; 58; 62; 65], **q** = [350; 375; 425; 450]

α	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
L(α)	12.250	12.443	12.638	12.834	13.032	13.231	13.432	13.634	13.838	14.043	14.250
R(α)	20.250	20.003	19.758	19.514	19.272	19.031	18.792	18.554	18.318	18.083	17.850

Table 11: Uncertain profit $\boldsymbol{\pi}$ (7) by global optimization, trapezoidal fuzzy input **p** = [55; 58; 62; 65]

α	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
L(α)	15.750	15.779	15.806	15.832	15.856	15.877	15.898	15.916	15.932	15.947	15.960
R(α)	16.000	16.000	16.000	16.000	16.000	16.000	16.000	16.000	16.000	16.000	16.000

4.3 Simple Case with Trapezoidal Fuzzy Numbers

The preceding calculations have been redone with trapezoidal input parameters according to (37). Table 10 shows the result with **p** = [55; 58; 62; 65], **q** = [350; 375; 425; 450] and Table 11 with **p** = [55; 58; 62; 65]. The interpretation of the resulting trapezoidal profit in Table 10 is that the most possible values corresponding to $\alpha = 1$ are in the interval [14.250; 17.850] and values outside the interval [12.250; 20.250] are impossible (unchanged from Table 8). The possibility of attaining a particular profit value is given by the value of the membership function. Arguments along the same lines hold for the results in Table 11. Again, the maximum profit of 16.000 has been correctly calculated for all values of α.

5 A Railway Reconstruction Project

Consider the case of estimating the total cost incurred by a railway reconstruction project described by independent fuzzy input variables, namely 18 cost items $\mathbf{X}_1, ..., \mathbf{X}_{18}$ and 3 correction factors $\mathbf{X}_{19}, ..., \mathbf{X}_{21}$. This case has been treated previously by using the concept of imprecise stochastic variables [23]. The correction factors are introduced in order to account for overall influences not accounted for by the individual cost items. The total cost before corrections is the sum $\mathbf{Y}_1 = \mathbf{X}_1 + \mathbf{X}_2 + ... + \mathbf{X}_{18}$. The total cost after corrections $\mathbf{Y}$ is a function of all 21 uncertain variables $\mathbf{Y} = (\mathbf{X}_1 + \mathbf{X}_2 + ... + \mathbf{X}_{18}) \cdot \mathbf{X}_{19} \cdot \mathbf{X}_{20} \cdot \mathbf{X}_{21}$.

Table 12: Cost estimation for railway reconstruction case by triangular fuzzy numbers

Var.	**Code**	**Item**	**[a; c; b]**
	0,00	**Management and specs.**	[1.732; 1.780; 1.884]
$\mathbf{X}_1$	0,10	Project management	[524; 540; 575]
$\mathbf{X}_2$	0,20	Construction management etc.	[975; 1.000; 1.050]
$\mathbf{X}_3$	0,30	Design specifications etc.	[233; 240; 259]
$\mathbf{X}_4$	**10,00**	**Environmental and soil eng.**	[864; 888; 950]
$\mathbf{X}_5$	**20,00**	**Traffic tasks**	[48; 50; 53]
	30,00	**Renewal of tracks**	[8.905; 9.190; 9.664]
$\mathbf{X}_6$	30,10	New outbound main track	[975; 1.000; 1.050]
$\mathbf{X}_7$	30,20	Track renewal at platform 3/5	[5.432; 5.600; 5.880]
$\mathbf{X}_8$	30,30	New platform edge	[1.533; 1.580; 1.643]
$\mathbf{X}_9$	30,40	Track renewal depot, West	[285; 300; 321]
$\mathbf{X}_{10}$	30,50	Track layout design	[682; 710; 770]
$\mathbf{X}_{11}$	**40,00**	**Platform and station**	[538; 560; 602]
$\mathbf{X}_{12}$	**50,00**	**Safety and signal installations**	[5.035; 5.245; 5.586]
	60,00	**Informatics incl. power supply**	[2.374; 2.417; 2.626]
$\mathbf{X}_{13}$	60,10	Phase 2-4	[78; 80; 86]
$\mathbf{X}_{14}$	60,20	Sub project management	[249; 259; 275]
$\mathbf{X}_{15}$	60,30	Passenger information	[1.009; 1.030; 1.123]
$\mathbf{X}_{16}$	60,40	Electrical power supply	[1.038; 1.048; 1.142]
	70,00	**Overhead line incl. pylons**	[3.507; 3.624; 3.787]
$\mathbf{X}_{17}$	70,10	Overhead cables	[3.021; 3.122; 3.262]
$\mathbf{X}_{18}$	70,20	Layout and planning	[486; 502; 525]
$\mathbf{Y}_1$		**Total cost before corrections**	[23.003; 23.754; 25.152]
$\mathbf{X}_{19}$	A	Internal decision process	[1,006; 1,032; 1,098]
$\mathbf{X}_{20}$	B	Design specifications etc.	[1,009; 1,040; 1,100]
$\mathbf{X}_{21}$	C	Working process	[1,021; 1,042; 1,084]
$\mathbf{Y}$		**Total cost after corrections**	[23.842; 26.565; 32.930]

Railway experts with relevant project experience should estimate the uncertain input variables. Subsequently, the total cost $\mathbf{Y}$ is calculated by means of global optimization. The total cost estimation results are shown in Table 12 in the case of triangular fuzzy input variables. The term “code” refers to the cost structure hierarchy. In Table 13, the membership function of total cost $\mathbf{Y}$ after corrections is shown.

For α = 1 the single point calculation of 26.565 is shown corresponding to an ordinary calculation without uncertainty. This value is the most possible one and is attained when all input variables are attaining their most possible values. For α = 0 the total cost after corrections is within the interval [23.842; 32.930] and any value outside this interval is impossible considering the uncertain input variables.

Likewise, Table 14 contains the results using trapezoidal fuzzy input variables and Table 15 the membership function for the total cost after corrections. For α = 1 the most possible values will be in the interval [25.347; 29.029] corresponding to all input variables to be confined within their inner intervals. For α = 0 the total cost after corrections is within the interval [23.842; 32.930], as was the case with triangular fuzzy input variables because the outer limits of the trapezoidal numbers are identical to those of the triangular input variables.

Table 13: Total cost after corrections for railway reconstruction case by triangular fuzzy number

α	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
L(α)	23.842	24.108	24.353	24.624	24.875	25.148	25.480	25.734	26.020	26.276	26.565
R(α)	32.930	32.244	31.592	30.895	30.269	29.626	28.983	28.387	27.743	27.161	26.565

Table 14: Cost estimation for railway reconstruction case by trapezoidal fuzzy numbers

Var.	Code	Item	[a; c; d; b]
	0,00	**Management and specs.**	[1.732; 1.757; 1.820; 1.884]
X_1	0,10	Project management	[524; 534; 551; 575]
X_2	0,20	Construction management etc.	[975; 985; 1.025; 1.050]
X_3	0,30	Design specifications etc.	[233; 238; 244; 259]
X_4	**10,00**	**Environmental and soil eng.**	[864; 872; 910; 950]
X_5	**20,00**	**Traffic tasks**	[48; 49; 51; 53]
	30,00	**Renewal of tracks**	[8.907; 9.109; 9.374; 9.664]
X_6	30,10	New outbound main track	[975; 985; 1.015; 1.050]
X_7	30,20	Track renewal at platform 3/5	[5.432; 5.556; 5.735; 5.880]
X_8	30,30	New platform edge	[1.533; 1.572; 1.599; 1.643]
X_9	30,40	Track renewal depot, West	[285; 295; 310; 321]
X_{10}	30,50	Track layout design	[682; 701; 715; 770]
X_{11}	**40,00**	**Platform and station**	[538; 556; 575; 602]
X_{12}	**50,00**	**Safety and signal installations**	[5.035; 5.176; 5.345; 5.586]
	60,00	**Informatics incl. power supply**	[2.374; 2.398; 2.473; 2.626]
X_{13}	60,10	Phase 2-4	[78; 79; 83; 86]
X_{14}	60,20	Sub project management	[249; 253; 262; 275]
X_{15}	60,30	Passenger information	[1.009; 1.025; 1.073; 1.123]
X_{16}	60,40	Electrical power supply	[1.038; 1.041; 1.055; 1.142]
	70,00	**Overhead line incl. pylons**	[3.507; 3.598; 3.759; 3.787]
X_{17}	70,10	Overhead cables	[3.021; 3.100; 3.250; 3.262]
X_{18}	70,20	Layout and planning	[486; 498; 509; 525]
Y_1		**Total cost before corrections**	[23.005; 23.515; 24.307; 25.152]
X_{19}	A	Internal decision process	[1,006; 1,021; 1,055; 1,098]
X_{20}	B	Design specifications etc.	[1,009; 1,022; 1,073; 1,100]
X_{21}	C	Working process	[1,021; 1,033; 1,055; 1,084]
Y		**Total cost after corrections**	[23.842; 25.347; 29.029; 32.930]

Table 15: Total cost after corrections for railway reconstruction case by trapezoidal fuzzy number

α	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
L(α)	23.842	23.964	24.138	24.285	24.436	24.557	24.737	24.864	25.039	25.193	25.347
R(α)	32.930	32.525	32.116	31.716	31.314	30.984	30.557	30.173	29.785	29.408	29.029

6 Comparison with Monte Carlo Simulation Approach

In order to compare the approach using triangular fuzzy numbers with the probability approach the cost estimation of the railway reconstruction case in Section 5 is used. The triangular fuzzy input variables are interpreted as triangular probability distributions and used as input variables in a Monte Carlo simulation. In this way numerically identical input variables from Table 12 are used to contrast fuzzy representation of imprecision with probabilistic representation of variation. The Monte Carlo simulation program [18], was run with 10.000 iterations and the output result for total cost after corrections fitted with a normal distribution of mean value 27.590 and standard deviation 822. This is shown in Fig. 1 together with the corresponding triangular fuzzy number result of Table 13. Not surprisingly there is a substantial numerical difference between the results. When the input variables are interpreted as probability distributions the resulting total cost are much narrower compared to the triangular fuzzy representation, the latter clearly displaying the possibility of a much broader range of outcomes. Further, notice that the triangular fuzzy result at $\alpha = 1$ substantially differs from the mode and mean value of the normal distribution, the former maintaining the original skewness of the uncertain input variables. Finally, the result from the Monte Carlo simulation gives a minimum and maximum value of the probability distribution of respectively 25.030 and 30.819, which should have been 23.842 and 32.930 as is seen from the triangular fuzzy result. This observation indicates that one should not rely on Monte Carlo simulations when the tales of the distributions are important. This is a confirmation of previously established results in a comprehensive comparative study of alternative approaches to modelling of economic uncertainty [22].

Similarly, trapezoidal probability distributions based on the data from Table 14 have been used in a Monte Carlo simulation resulting in a normal distribution with mean value 27.700 and standard deviation 879. Not surprisingly, this result is very close to the one obtained above using triangular distributions.

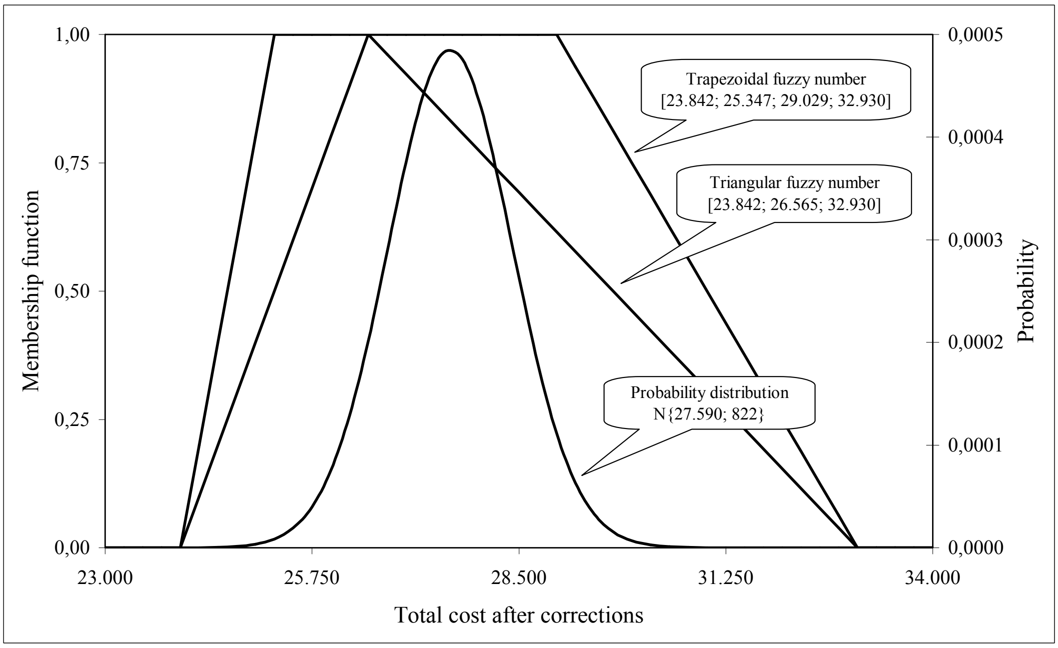

Figure 1: Cost estimation for railway reconstruction case. Probability distribution by Monte Carlo simulation with triangular input variables based on data from Table 12. Triangular fuzzy number from Table 13. Trapezoidal fuzzy number from Table 15

7 Conclusions

In order to calculate correct results with intervals and fuzzy numbers global optimization techniques must be implemented in contrast to straightforward application of interval arithmetic. The experimental calculations reported in the paper also show that further care should be taken not to introduce intermediate variables, e.g. when variables appear more than once, which might also introduce excess width intervals. Not only uncertain input variables have been considered but also an uncertain model based on uncertain model parameters.

The results of the paper indicate that correct calculations of intervals (and fuzzy number membership functions) allow for rather strong statements pertaining to economic uncertainties. For example, it can be said that provided all uncertain input variables stay within their limits the uncertain output variables will stay within their limits. Obviously, it is an advantage of the interval, triangular fuzzy, and trapezoidal fuzzy representation of uncertainty that the meaning is easily communicated and understood.

The substantial difference between uncertainty results produced by the possibilistic approach and the probabilistic approach has been demonstrated by interpreting and processing numerically identical uncertain input data to the railway reconstruction project in two different ways. Firstly, triangular input data were interpreted as triangular fuzzy numbers and processed accordingly by global optimization. Secondly, they were interpreted as triangular probability distributions and processed by Monte Carlo simulation. Not surprisingly, the possibilistic approach resulted in a substantially wider range of uncertainty reflecting the non-statistical nature of the uncertainty. Using trapezoidally shaped input data yielded a similar result. These results usually create heated discussions with statisticians who often claim that it proves the probabilistic approach to be superior and more realistic because it produces a smaller range of uncertainty. However, the point is that the two approaches allow for different kind of arguments. Basically, if the uncertainty is of a statistical nature, statistical methods are recommended. If the uncertainty is of a non-statistical nature, non-statistical methods should be used.

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